Project MATHEMATICS!

Program Guide and Workbook

to accompany the videotape on

The Story of π

Project MATHEMATICS!
CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, CA 91125 • (818) 356-6345

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to accompany the videotape on

The Story of π

Written by TOM M. APOSTOL, California Institute of Technology

with the assistance of the

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THE STORY OF PI

was produced by Project MATHEMATICS!

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AIMS AND GOALS OF *Project MATHEMATICS!*

Project MATHEMATICS! uses computer-animated videotapes to show students that learning mathematics can be exciting and intellectually rewarding. The videotapes treat mathematical concepts in ways that cannot be done at the chalkboard or in a textbook. They provide an audiovisual resource to be used together with textbooks and classroom instruction. Each videotape is accompanied by a workbook designed to help instructors integrate the videotape with traditional classroom activities. Video makes it possible to transmit a large amount of information in a relatively short time. Consequently, it is not expected that all students will understand and absorb all the information in one viewing. The viewer is encouraged to take advantage of video technology that makes it possible to stop the tape and repeat portions as needed.

The manner in which the videotape is used in the classroom will depend on the ability and background of the students and on the extent of teacher involvement. Some students will be able to watch the tape and learn much of the material without the help of an instructor. However, most students cannot learn mathematics by simply watching television any more than they can by simply listening to a classroom lecture or reading a textbook. For them, interaction with a teacher is essential to learning. The videotapes and workbooks are designed to stimulate discussion and encourage such interaction.

STRUCTURE OF THE WORKBOOK

The workbook begins with a brief outline of the video program, followed by suggestions of what the teacher can do before showing the tape. Numbered sections of the workbook correspond to capsule subdivisions in the tape. Each section summarizes important points in the capsule. Some sections contain exercises that can be used to strengthen understanding. The exercises emphasize key ideas, words and phrases, as well as applications. Some sections suggest special projects that students can do for themselves.

I. BRIEF OUTLINE OF THE PROGRAM

The videotape begins with a brief *Review of Prerequisites* explaining how perimeters and areas of plane figures change when the figures are expanded or contracted. This topic is treated more thoroughly in a separate program on similarity. If the students have seen the program on similarity, this section will be a review. If not, some time should be devoted to the introductory section of this workbook, which discusses basic properties of similar figures.

The program itself opens with a reporter interviewing several young people, asking "What can you tell me about the number pi?" Each person gives a different answer, some of which are only partially correct. Later in the program the same people respond with correct answers.

The program defines pi as the ratio of the circumference to the diameter of a circle, then explains that pi appears in a variety of formulas, many of which having nothing to do with circles. After discussing the early history of pi, the program invokes similarity to explain why the ratio of circumference to diameter is the same for all circles, regardless of their size. This ratio, a fundamental constant of nature, is denoted by the Greek letter π. The formula $2\pi r$ for the circumference of a circle of radius r follows from the definition of π.

The program then turns to another ratio that is the same for all circles: the area of a circular disk divided by the square of its radius. By showing that a circular disk of radius r has area πr^2, Archimedes proved that this constant ratio is also equal to π.

Two animated proofs of the area formula are given. This is followed by the Archimedes method for estimating π by comparing the circumference of a circle with the perimeters of inscribed and circumscribed polygons.

The next segment describes a sequence of improved estimates for the value of π, and points out that π is irrational (not the quotient of two integers). After demonstrating the appearance of π in probability problems, the program returns briefly to the reporter who interviews the students again, asking, "*Now* what can you tell me about pi?" This time, each student gives a different correct statement about π. The concluding segment explains that major improvements in the estimates for π represent landmarks of important advances in the history of mathematics.

II. BEFORE WATCHING THE VIDEOTAPE

The first capsule of the video, entitled *Review of Prerequisites*, mentions some basic properties of similar figures that are used in this program. If students are familiar with these ideas, this section will serve as a review. If not, an effort should be made to acquaint them with these ideas before viewing the tape. A good way to do this is to have the students read the next section and solve the exercises.

KEY WORDS AND STATEMENTS:

Scaling of a plane figure.

Circumference of a circle.

Area of a circular disk.

Ratio of two numbers.

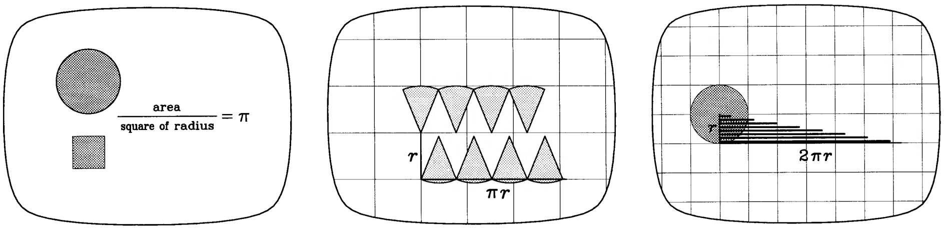
THE MAIN IDEAS IN THIS PROGRAM:

The ratio of the circumference of a circle to its diameter is the same for all circles, regardless of their size. This ratio, a fundamental constant of nature, is denoted by the Greek letter π (written pi in English and pronounced pie, as in 'apple pie'). Consequently, the circumference of a circle of diameter d is equal to πd, or $2\pi r$, where r is the radius.

The number π appears in many formulas for calculating areas and volumes of circular objects. For example, the area of a circular disk of radius r is equal to πr^2, and the volume of a sphere of radius r is $4\pi r^3/3$. The number π also appears in formulas that have nothing to do with circles. Examples occur in engineering, in planetary science, and in probability problems.

Like any irrational number, π can be approximated to any degree of accuracy by rational numbers. Two commonly used approximations are 22/7, first obtained by Archimedes in the third century B.C., and 355/113, found in the fifth century by the Chinese mathematician Tsu Chung-Chi.

III. TABLE OF CONTENTS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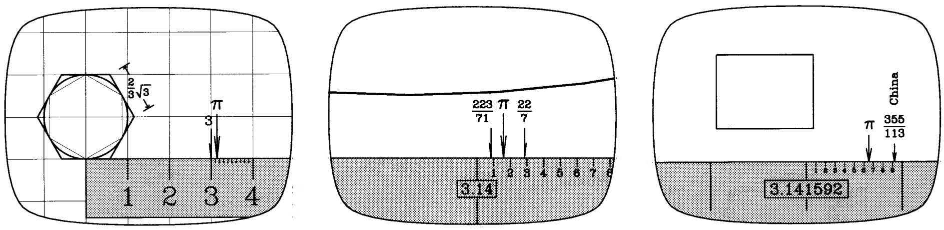
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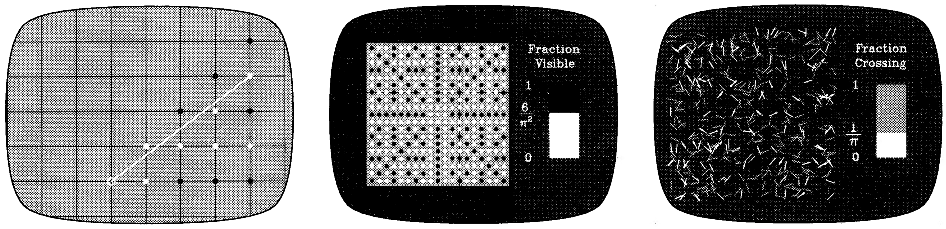
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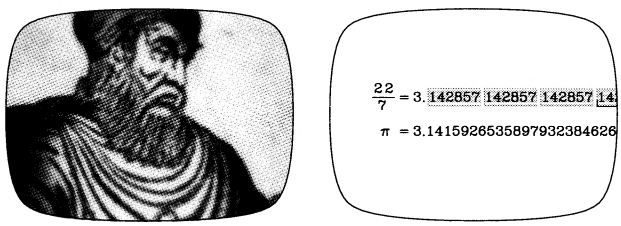
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Review of Prerequisites

Similar triangles

Similarity of triangles is a simple concept that is used again and again in many applications of geometry. Figure 1 shows two similar triangles. They have the same shape, because corresponding angles are equal, but are of different size, with lengths of corresponding sides being proportional.

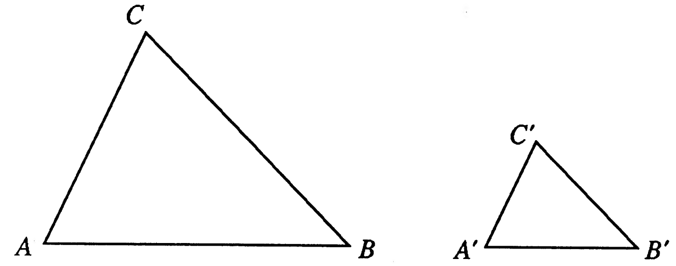


Figure 1. Similar triangles.

In Figure 1, the sides of the larger triangle are twice as long as those of the smaller, so

$$AB = 2A'B', \quad BC = 2B'C', \text{ and } AC = 2A'C' .$$

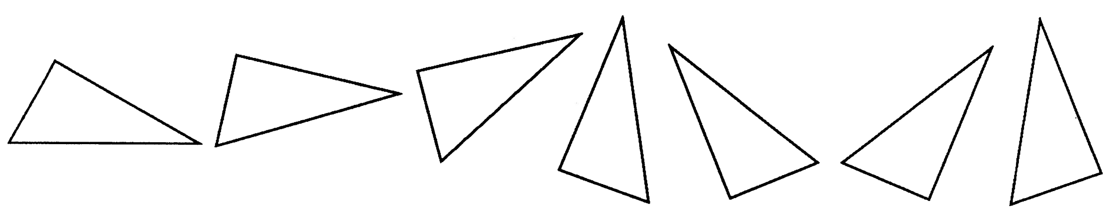
In general, two triangles ABC and $A'B'C'$ are called *similar* if corresponding angles are equal,

$$\angle A = \angle A', \quad \angle B = \angle B', \quad \angle C = \angle C'.$$

This implies that the lengths of corresponding sides have the same ratio: $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$.

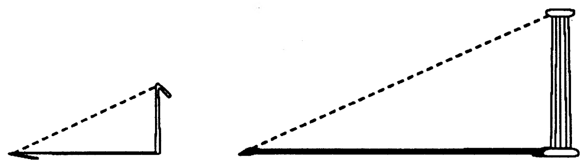
In Figure 1 this common ratio is equal to 2. For arbitrary similar triangles, the common ratio is some positive number r, and we say that triangle ABC is similar to triangle $A'B'C'$ with *similarity ratio* or *scaling factor* r. The number r is called an *expansion factor* if r is greater than 1 (written in symbols as $r > 1$), and a *contraction factor* if r is less than 1 (written $r < 1$). For example, in Figure 1, the lengths of the sides of triangle ABC are twice as long as those of triangle $A'B'C'$, so ABC is obtained from $A'B'C'$ by expansion by the factor $r = 2$. On the other hand, triangle $A'B'C'$ is obtained from ABC by contraction by the factor $r = 1/2$. Expansion or contraction of the sides of a triangle without distortion of the angles always produces a similar triangle. When $r = 1$, the two triangles have the same shape *and* size, and are called *congruent.*

In summary, similar triangles have the same shape but may be of different size. If they have the same shape and size (similarity ratio 1), they are congruent. When a triangle is moved without changing the lengths of its sides, for example if it is shifted, rotated, or flipped over to form a mirror image, a congruent triangle is obtained. All these triangles are congruent and, of course, similar.



Exercises involving similar triangles

1. The Greek mathematician Thales in the 6th century B.C. is said to have invented a method for determining the height of a column by looking at its shadow and the shadow of his staff. Look at the figure and then explain how his method works.



2. If two triangles ABC and $A'B'C'$ have two of their corresponding angles equal, say $\angle A = \angle A'$ and $\angle B = \angle B'$, show that the third angles are also equal: $\angle C = \angle C'$.

3. Given two similar triangles ABC and $A'B'C'$ with similarity ratio r. (An example with $r = 2$ is shown in Figure 2.) Let CP and $C'P'$ denote the altitudes of the two triangles. (The segment CP is perpendicular to the base AB, and $C'P'$ is perpendicular to base $A'B'$.)

(a) Prove that triangles APC and $A'P'C'$ are also similar with similarity ratio r. This implies that the ratio of the altitudes $CP/C'P'$ is also equal to the similarity ratio r.

(b) Do the same for medians and angle bisectors of similar triangles.

(c) Look at the results of (a) and (b) and make a general conjecture.

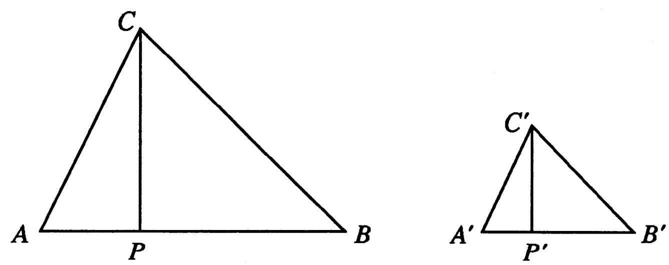
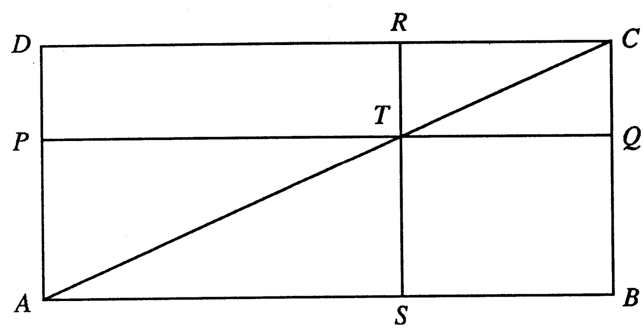
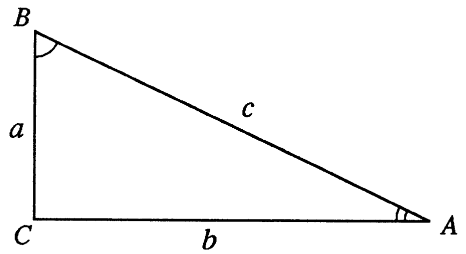


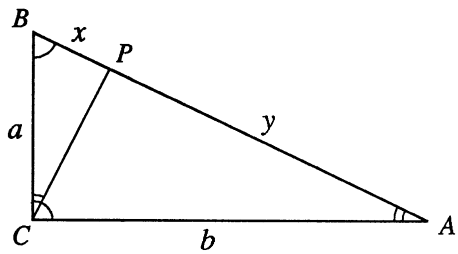
Figure 2. Corresponding altitudes of similar triangles have the same ratio as corresponding sides.

4. In the following figure, $ABCD$ is a rectangle, PQ is parallel to AB and intersects RS, which is parallel to DA, at a point T on the diagonal AC. Prove that the three triangles AST, TQC, and ABC are similar.



5. This exercise outlines a proof of the Theorem of Pythagoras based on similar triangles. Right triangle ABC has legs of length a and b, and hypotenuse of length c. Line CP is constructed perpendicular to the hypotenuse AB, forming two new right triangles APC and BPC, and dividing the hypotenuse into segments of length x and y, as shown.





(a) Prove that triangles ABC and CBP are similar, and that $x/a = a/c$.

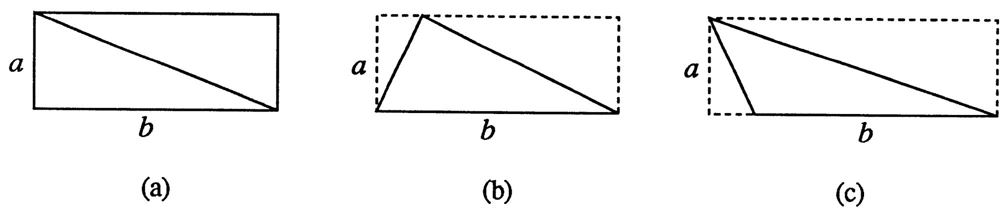
(b) Prove that triangles ABC and ACP are similar, and that $y/b = b/c$.

(c) Part (a) shows that $a^2 = cx$, whereas part (b) shows that $b^2 = cy$. Add these equations to deduce the Theorem of Pythagoras:

$$a^2 + b^2 = c^2.$$

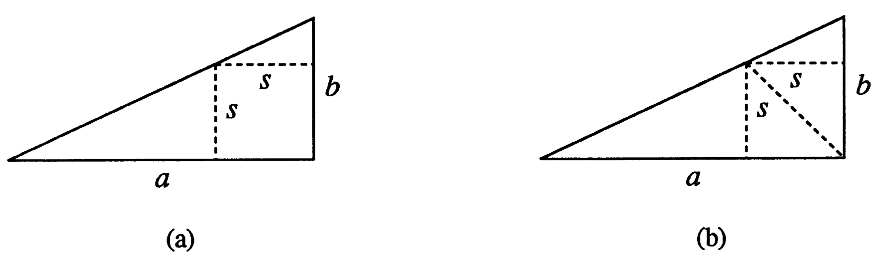
Exercises involving areas of triangles

6. This exercise shows how the formula for the area of a triangle can be deduced from the formula for the area of a rectangle. A rectangle of base b and altitude a has area equal to the product ba. A diagonal divides the rectangle into two congruent right triangles (shown below in (a)), each with area $ba/2$. By comparing an arbitrary triangle of base b and altitude a to two right triangles, as shown in (b) or (c), prove that its area is also equal to $ba/2$.

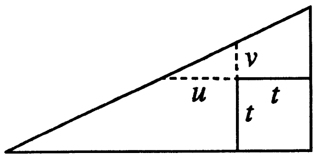


7. (a) A square of side s is inscribed in a right triangle with legs a and b as shown in (a). Use similar triangles to show that $s/(a - s) = (b - s)/s$. Solve this equation for s to obtain $s = ab/(a + b)$.

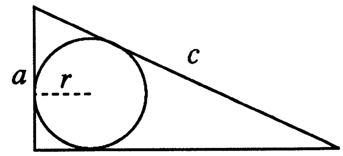
(b) Refer to the diagram in (b) and use an argument based on areas to find another derivation of the formula for s in part (a).



(c) Generalize the problem as follows. Draw a square of side t and let u and v be the lengths of the dotted lines shown in the figure:



Exercise 7 (c).



Exercise 8 (a).

Prove that $t = (ab - uv)/(a + b + u + v)$.

8. (a) A circle of radius r is inscribed in a right triangle with legs of length a and b and hypotenuse c, as shown above. Use an argument based on area to show that $r = ab/(a + b + c)$.

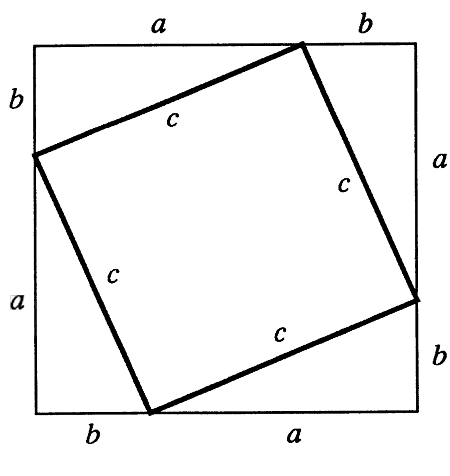
(b) Generalize the argument to show that if a, b, c are the sides of *any* triangle, then the radius r of the inscribed circle is given by $r = 2K/(a + b + c)$, where K is the area of the triangle.

9. Early Chinese documents contain a proof of the Theorem of Pythagoras based on the diagram below. Essentially the same proof was discovered by U. S. Congressman James A. Garfield, a few years before he became the 20th president of the United States.

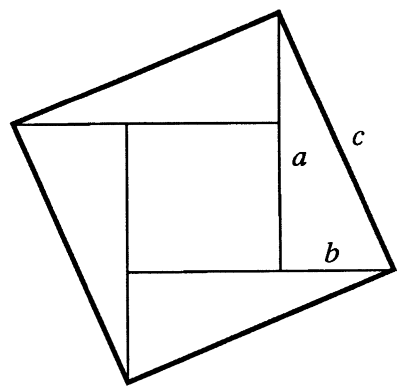
(a) Start with a square of edge $a + b$ and cut off four right triangles at the corners, each with hypotenuse c, and legs a and b, as shown in (a). Prove that the inner figure is a square.

(b) The area of the inner square, c^2, plus 4 times the area of each triangle, $ab/2$, is equal to the area of the larger square, $(a + b)^2$. Show that this equality of areas implies the Theorem of Pythagoras:

$$a^2 + b^2 = c^2.$$



(a)



(c)

(c) In a similar vein, prove the Theorem of Pythagoras by placing the right triangles *inside* the square of edge c as shown in (c).

BEHAVIOR OF PERIMETERS AND AREAS OF PLANE FIGURES UNDER SCALING

Perimeters of similar figures

The perimeter of a triangle is the sum of the lengths of its sides. If each side of a triangle is multiplied by a factor r, the perimeter of the triangle is also multiplied by r. This can be stated as follows:

Scaling a triangle by a factor r multiplies its perimeter by r.

The concept of similarity also applies to more general figures. For example, if a polygon P is composed of several triangles, and if each triangle is expanded or contracted by the same factor r, then the new polygon P' so obtained is said to be similar to P with similarity ratio r. The perimeter of P' is r times the perimeter of P.

Scaling a polygon by a factor r multiplies its perimeter by r.

The same is true for a circle. Scaling a circle by a factor r multiplies its radius and perimeter by the same factor r. The perimeter of a circle is also called its *circumference*.

Behavior of area under expansion or contraction

If two triangles are similar with similarity ratio r, lengths of corresponding sides have ratio r, and the same is true for the ratios of corresponding altitudes. (See Exercise 3 in the foregoing section.) Because the area of a triangle is one-half base times altitude, if the base and altitude are each multiplied by r, the area is multiplied by r^2. Therefore, if two triangles are similar with similarity ratio r, the ratio of their areas is r^2. This can be stated as follows:

Scaling a triangle by a factor r multiplies its area by r^2.

A general polygonal figure can be decomposed into triangular pieces. An example is shown in Figure 3. If we let A denote the area of the polygonal figure and let $A_1, A_2, ..., A_5$, denote the areas of the triangular pieces, then A is the sum of the areas of the triangular pieces:

$$A = A_1 + A_2 + A_3 + A_4 + A_5.$$

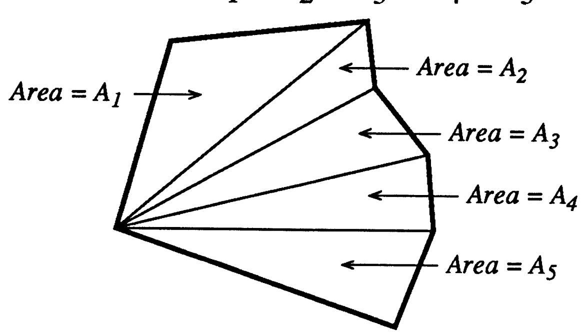


Figure 3. A polygonal figure decomposed into five triangular pieces.

If the polygonal figure is scaled by a factor r, each triangular piece is scaled by the same factor, and the area of each triangular piece gets multiplied by the factor r^2. Thus the scaled polygon has area

$$r^2A_1 + r^2A_2 + r^2A_3 + r^2A_4 + r^2A_5.$$

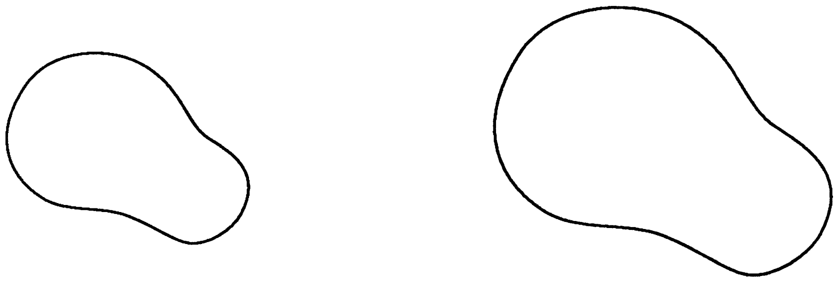
The common factor r^2 can be factored out and we get

$$\text{area of scaled polygon} = r^2(A_1 + A_2 + A_3 + A_4 + A_5) = r^2A.$$

The same argument works for any polygon.

Scaling a polygon by a factor r multiplies its area by the factor r^2.

Now take a more general plane figure F with curved boundaries, and let S denote its area. Scaling by a factor r produces a similar figure F', as shown by the example in Figure 4.



Original figure F
area = S

Scaled figure F'
area = r^2S

Figure 4. Scaling a plane figure by a factor r multiplies its area by r^2.

The area of F' is equal to r^2S. This is because F can be approximated from the inside and from the outside by polygons, as shown in Figure 5. The area of F lies between the areas of the inner and outer polygons:

$$\text{area of inner polygon} < S < \text{area of outer polygon.}$$

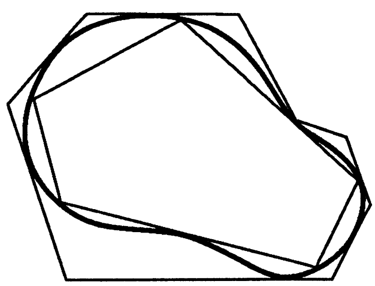


Figure 5. A curved shape approximated by inner and outer polygons.

By taking approximating polygons with an increasing number of sides, the areas of the inner and outer polygons can be made arbitrarily close to each other and therefore arbitrarily close to the area of F. If the entire diagram is scaled by a factor r, the approximating polygons are also scaled by a

factor r and their areas are multiplied by the factor r^2:

$$r^2(\text{area of inner polygon}) < \text{area of } F' < r^2(\text{area of outer polygon}).$$

Because the areas of the inner and outer polygons can be made arbitrarily close to S, it follows that the area of F' is equal to r^2S. [This is an application of the 'squeezing principle' in the theory of limits. If a fixed number a lies between two changing quantities x and y, and if both x and y approach the same limit z, then $a = z$.] Therefore we have the following property:

Scaling a plane figure by a factor r multiplies its area by r^2.

Circumference and area of a circular disk

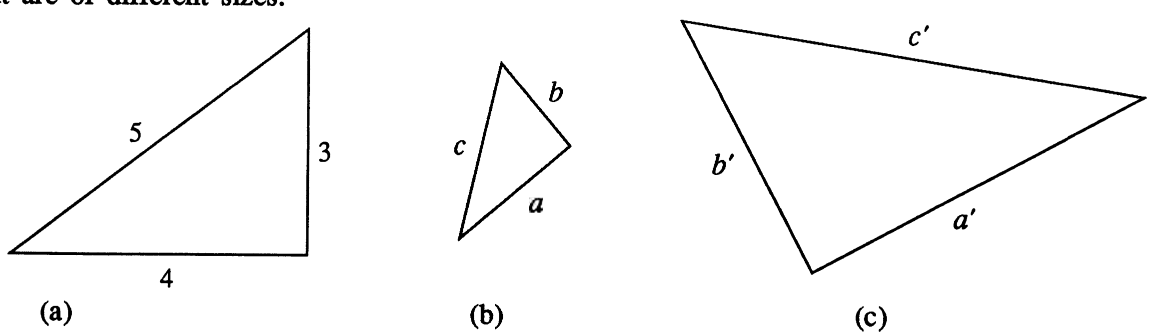
The foregoing ideas can be applied to circles. Any two circles are similar. A circle of radius r is similar to a unit circle (a circle of radius 1), the similarity ratio being equal to r. (The number 1 here represents the unit of measure being used. It could be 1 inch, 1 foot, 1 meter, 1 centimeter, or any other convenient unit of distance. The radius r is measured relative to the same unit.)

A circle, together with all the points inside the circle, is called a *circular disk*. If the unit circular disk has circumference C and area A, then the disk of radius r has circumference Cr (because lengths get multiplied by r) and area Ar^2 (because areas get multiplied by r^2). This program will show that $C = 2\pi$ and that $A = \pi$.

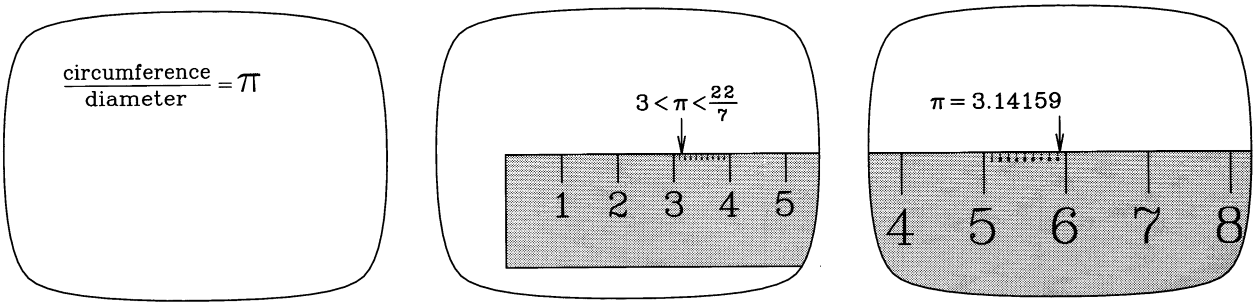
APPENDIX: UNIT LENGTH AND UNIT AREA (optional)

Commerce and science share a common need for standardized units of measure. Throughout history, governments and scientific organizations have tried to establish standard units of time, length, area, and weight. Some of the common units of length are 1 inch, 1 foot, 1 mile, 1 centimeter, 1 meter, 1 kilometer. Once the unit of length is chosen, the unit of area is taken to be a square with edges of unit length. This is called a unit square, and its area is said to be 1 square inch, 1 square foot, 1 square mile, etc. For example, a rectangle with edges a and b measured in inches is said to have area ab square inches.

In geometry, lengths are often specified without reference to the choice of unit length. For example, if a line segment is said to have length 3, it is understood that the segment is 3 times as long as the unit segment. Once a choice of units has been made, two perpendicular line segments having lengths 3 and 4 determine a right triangle with hypotenuse of length 5. In the right triangle in (a), the unit of length is 1 centimeter. Every 3-4-5 right triangle is similar to this one. Two examples are shown in (b) and (c), one smaller and one larger. All three triangles have the same shape but are of different sizes.



1. *What is the number π?*



1. Choose several circular objects (for example, jar lids, pots and pans, hula hoops, bicycle wheels) whose circumference C and diameter d can be conveniently measured with a tape measure. For each object, record the measured values of C and d, and calculate the ratio of these measurements, C/d.

(a) Explain why the calculated values C/d may differ slightly for objects of different sizes, even though the theory of similarity says that this ratio is independent of the radius.

(b) If different people measure the same objects, it is likely that their tabulated values will not be identical. What do you think are the reasons for these differences?

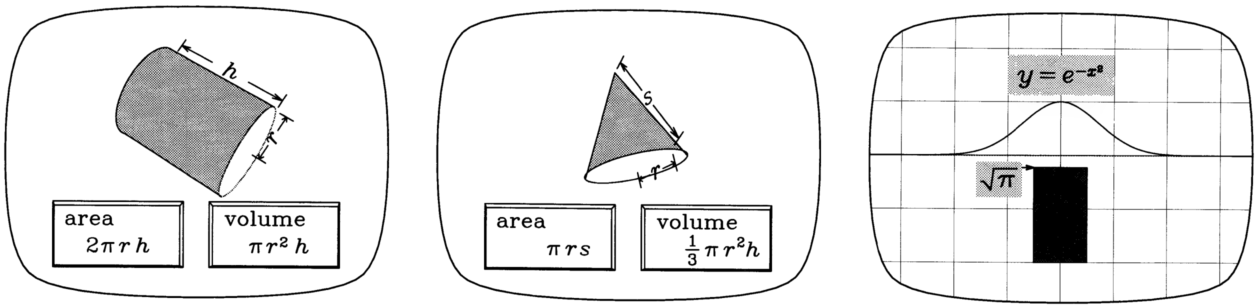
2. Pretend that you are in ancient Egypt around 3,000 B.C. and want to estimate the value of π. Your only tools are wooden stakes and ropes. You have no compass, no pencil or paper, no calibrated measuring tape, and no arithmetic as we know it today (with Arabic numerals and decimals). Assume you can find a flat patch of moist sand along the banks of the Nile. Explain how you would draw a circle in the sand, how you would measure its circumference and diameter, and how you would determine a value for the ratio of circumference to diameter.

The history of π as a symbol

Although people have been interested in the circle ratio for more than four thousand years, the use of the Greek letter π as the symbol designating this ratio is relatively recent. The Englishman William Jones (1675-1749) is generally recognized as the first to use the symbol π for this ratio. The symbol appeared in his book *Synopsis Palmariarum Matheseos*, published in 1706, which included 100 decimal places of π calculated by John Machin (1680-1752). Earlier, William Oughtred and Isaac Barrow (one of Isaac Newton's teachers) had used the same symbol to denote the circumference of a unit circle. A French calculus textbook dated 1777 also used π for the circumference of a unit circle.

The letters c (for circumference) and p (for perimeter) were sometimes used for the circle ratio, but the Greek letter π became widely accepted after Leonhard Euler used it in his famous *Introductio in Analysin Infinitorum*, published in 1748. It is generally believed that the letter π was chosen because it is the first letter of the Greek words for *perimeter* and *periphery*.

2. *Some uses of π*



The number π is defined to be the ratio of the circumference of a circle to its diameter. If C denotes the circumference and d the diameter, the definition states that $\pi = C/d$, and hence $C = \pi d$. This formula expresses the circumference in terms of the diameter. It is often written as

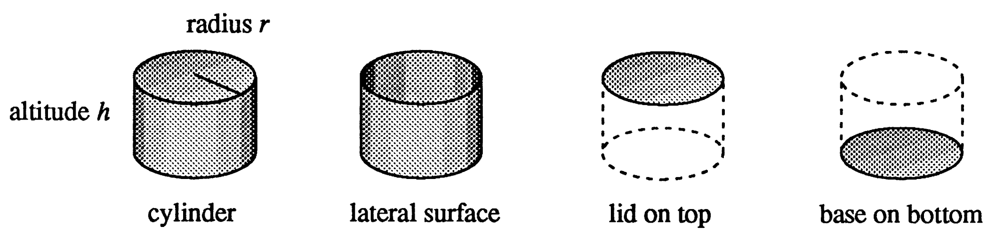
$$C = 2\pi r,$$

where r is the radius (half the diameter). In Section 4 of this workbook we will show that the formula for the circumference can be used to derive the formula

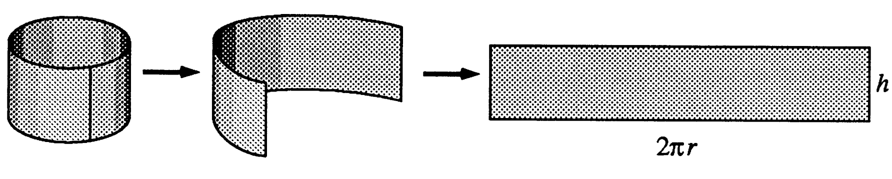
$$A = \pi r^2$$

for the area A of a circular disk of radius r.

Whenever you see a solid of rotation, such as a sphere, cylinder, cone, or torus, the number π is always there, in formulas for calculating areas and volumes. For example, suppose we want to determine the surface area of a cylindrical tin can of radius r and altitude h, shown below. Its surface consists of three parts, the *lateral* surface, plus two circular disks each having area πr^2, a lid on the top and a base on the bottom.



The lateral surface can be cut and unwrapped to form a rectangle with the same area, as shown here:

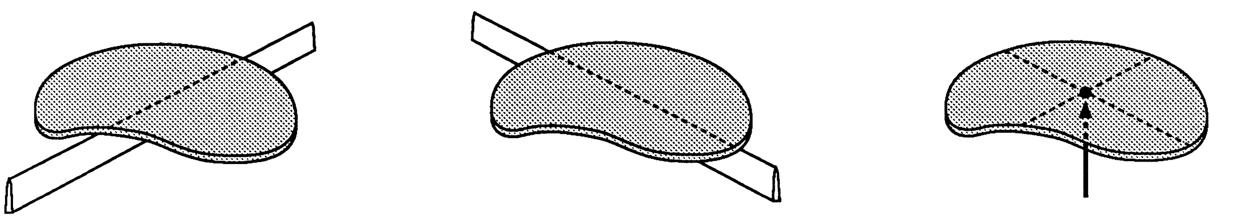


The rectangle has base $2\pi r$ (the circumference of the circle) and altitude h, so its area is $2\pi rh$. This is called the *lateral area* of the cylinder. The total surface area consists of the lateral area, $2\pi rh$, plus the areas of the two end pieces, each of area πr^2, for a total area of $2\pi rh + 2\pi r^2$.

As we will see in a moment, the volume of the cylinder is equal to the area of the base times the altitude: volume = $\pi r^2 h$.

Ever since the invention of the potter's wheel, solids of rotation have been objects of great interest. Besides the cylinder mentioned above, some simple solids of rotation are the sphere, the right circular cone, and the torus. In the 3rd century A.D., Pappus of Alexandria formulated some simple rules for calculating the volume and surface area of certain solids, including solids of rotation. These rules are theorems that can be proved by using integral calculus, but they were discovered long before the invention of calculus. They are discussed here because of their intrinsic interest and because they reveal how the number π enters into formulas for volume and surface area of round objects.

Pappus' rule for volume refers to a point called the *centroid* of a plane region. A general definition of centroid requires integral calculus. But for a simple figure, such as a polygonal region or a portion of a circular disk, the following physical description is adequate. A plane figure cut out of a piece of cardboard of uniform thickness can be balanced if suitably placed on the edge of a ruler as shown in figure (a). Let's call the line where the ruler intersects the cardboard a *balancing axis.* If the region has an axis of symmetry, this axis will be a balancing axis. There is, of course, more than one balancing axis, as shown by the example in figure (b). But all balancing axes intersect at one point, the *centroid* of the region, where, the region would balance on the point of a pin.



(a) A balancing axis (b) Another balancing axis (c) Axes intersect at the centroid

Pappus' rule for the volume of a solid:

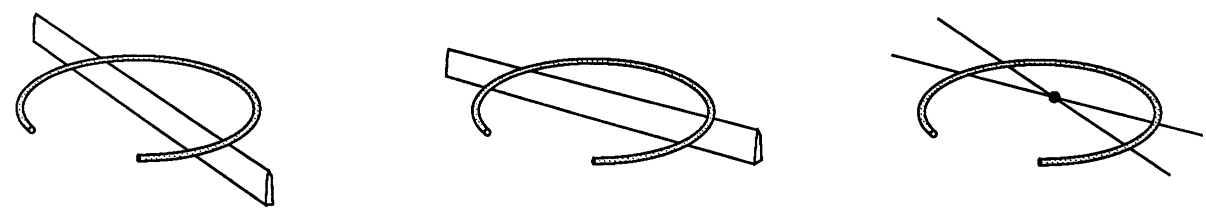
> *If a plane region moves to sweep out a solid, then the volume of the solid is equal to the area of the region multiplied by the length of the path of the centroid of the region.*

The motion in Pappus' rule is subject to certain restrictions. For example, the plane region must always be perpendicular to the direction of motion. This is the case for solids swept out by rotating a plane region about an axis in its plane, and for solids generated by moving a region along a line segment that is perpendicular to its plane. Also, the region is not allowed to sweep out part of the volume more than once. For solids of rotation this can be avoided by rotating about an axis in the plane of the region that does not intersect the region except possibly at boundary points.

For example, a circular disk of area πr^2 moving a distance h along a line as shown in (a), sweeps out a cylinder whose volume is $\pi r^2 h$, because h is the distance the centroid of the disk moves. The same cylinder can be generated in another way, by rotating a rectangle of base r and altitude h about an edge perpendicular to the base, as shown in (b). The centroid of the rectangle travels along a circle of radius $r/2$ through a distance $2\pi(r/2) = \pi r$. The product of this distance and the area of the rectangle rh is equal to $\pi r^2 h$, in agreement with the result in (a).

(a) Cylinder obtained by moving a circular disk. (b) Cylinder obtained by rotating a rectangle.

Pappus' rule for surface area refers to the centroid of a plane curve. Again, the general definition requires calculus, but the centroid can be visualized as follows. A curve made from a uniform wire can be balanced if suitably placed on the edge of a ruler. The line through the edge of the ruler is called a balancing axis, and all balancing axes intersect at the centroid of the curve.



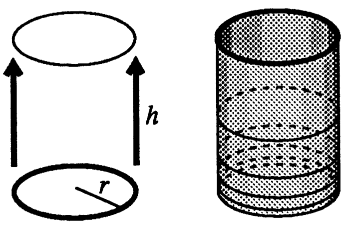
(a) A balancing axis (b) Another balancing axis (c) Axes intersect at the centroid

Pappus' rule for surface area:

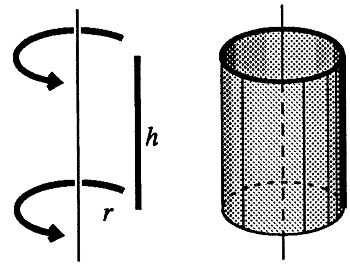
If a plane curve moves to sweep out a surface, then the area of the surface is equal to the length of the curve multiplied by the length of the path of the centroid of the curve.

Again, the motion in this rule is subject to restrictions. The plane of the curve must always be perpendicular to the direction of motion, and the curve is not allowed to sweep out part of the surface more than once.

For example, when a circle of radius r moves along a line segment of length h as shown below in (a) it sweeps out the lateral surface of a cylinder. By Pappus' rule the area swept out is equal to the product of the circumference, $2\pi r$, and the distance h the centroid moves. The result, $2\pi rh$, is the same as that obtained earlier by unwrapping a rectangle. The same surface can also be generated by rotating a line segment of length h about a parallel axis a distance r from the segment. The centroid of the segment moves a distance $2\pi r$, as shown in (b), and Pappus' rule states that the lateral surface area is equal to $2\pi rh$. Thus we have found the lateral area of cylinder by three different methods.



(a) Lateral surface of a cylinder obtained by moving a circle along a line segment.



(b) Lateral surface of a cylinder obtained by rotating a line segment.

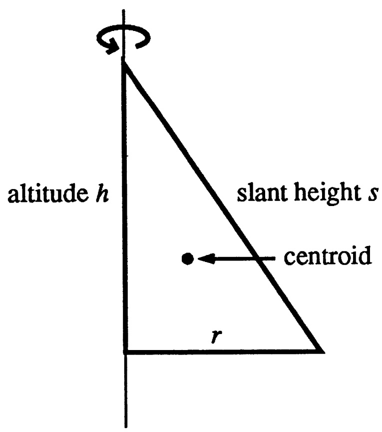
Exercises on areas and volumes of solids of rotation

1. A quarter of a circular disk of radius r is cut out of paper and rolled to form a right circular cone. Show that the altitude of the cone is $\sqrt{15}/4$ times r (nearly 97% of the radius r). What is the relation between the altitude and the radius if the cone is made from a semicircular disk?.

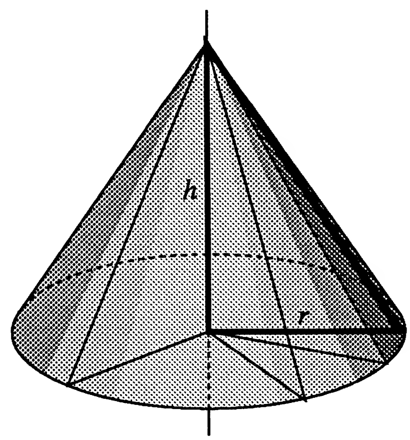
2. If a right triangle of base r and altitude h is rotated about its altitude it sweeps out a right circular cone. The length of the hypotenuse of the triangle is called the slant height of the cone and is denoted by s. The Theorem of Pythagoras shows that $s^2 = r^2 + h^2$. Use Pappus' rules to show that:

(a) The lateral surface area of the cone is πrs.

(b) The volume of the cone is $\pi r^2h/3$ (one-third the area of the base times the altitude). You may use the fact that the centroid of the right triangle shown is at a distance $h/3$ above the base and at a distance $r/3$ from the altitude.



Right triangle

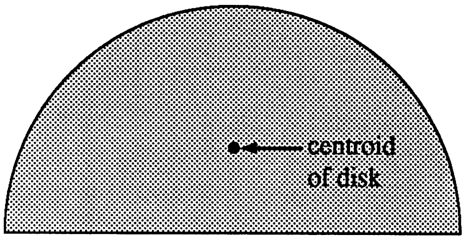


Right circular cone

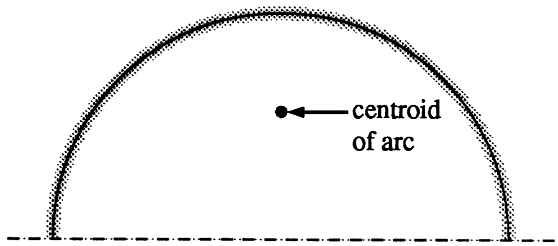
3. Rotate a semicircular disk of radius r about its diameter to sweep out a sphere of radius r. Use Pappus' rules to show that:

(a) The volume of the sphere is $4\pi r^3/3$. (You may use the fact from calculus that the centroid of the semicircular disk is at a distance $(4r)/(3\pi)$ above the diameter.) The volume of a sphere can be calculated by other methods, and then Pappus' rule can be used to determine the centroid of the semicircular disk.

(b) The surface area of the sphere is $4\pi r^2$. (You may use the fact from calculus that the centroid of a semicircular arc of radius r is at a distance $2r/\pi$ above its diameter.) The surface area of a sphere can be calculated by other methods, and then Pappus' rule can be used to determine the centroid of a semicircular arc.



Semicircular disk



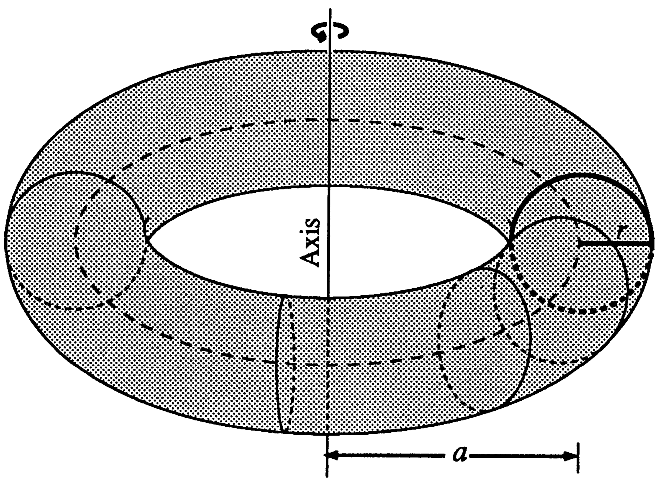
Semicircular arc

4. Rotate a circular disk of radius r about an axis in the plane of the disk at a distance $a > r$ from its center, to sweep out a *torus* (a doughnut-shaped figure). Use Pappus' rules to show that:

(a) The volume of the torus is $(2\pi a)(\pi r^2) = 2\pi^2 r^2 a$.

(b) The surface area of the torus is $(2\pi a)(2\pi r) = 4\pi^2 ar$.

(c) Do these results make sense without the restriction $a > r$?



Torus

Infinite series and products of ratios of integers that yield π

Although it is known that π is not a rational number (that is, π is not the ratio of two integers) there are many striking formulas that relate π to the integers. In 1656, John Wallis, Professor of Geometry at Oxford University, proved that $\pi/2$ is equal to an infinite product of rational numbers. The numerators of these fractions contain the even integers, each repeated twice, and the denominators contain the odd integers, each (except for 1) repeated twice. Wallis's product is written this way:

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots$$

This equation is to be interpreted as follows. Form a "partial product" by multiplying a finite number of factors. For example, the partial product of the first six factors is

$$\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7}.$$

Wallis proved that the limiting value of these partial products as the number of factors increases indefinitely is equal to $\pi/2$. This was the first formula to express π as the limit of a sequence of rational numbers.

A simpler formula, discovered by James Gregory in 1671, expresses $\pi/4$ as an infinite series of reciprocals of the odd integers with alternating signs:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots .$$

By analogy with infinite products, we consider a "partial sum" by adding a finite number of terms, and then find the limiting value of the partial sums as the number of terms increases indefinitely. Gregory proved that for this series the limit is $\pi/4$. The same result was discovered independently by Gottfried Wilhelm Leibniz in 1674, and the series is often called the Gregory-Leibniz series.

Another infinite series, this one for $\pi^2/6$, is the sum of the reciprocals of the squares of the positive integers:

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots .$$

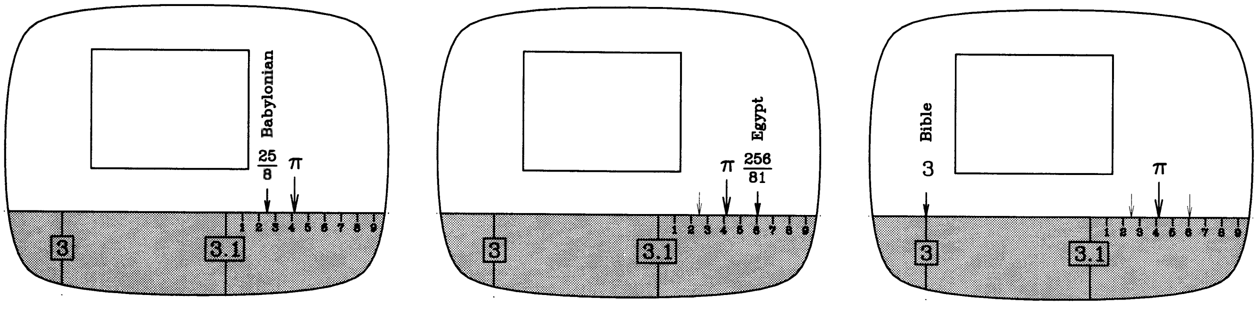
In 1736 Leonhard Euler showed that the sum of this series is $\pi^2/6$. He also showed that this series can be expressed as an infinite product involving all the prime numbers 2, 3, 5, 7, 11, 13, ... Specifically, he showed that

$$\frac{\pi^2}{6} = \frac{2^2}{2^2 - 1} \cdot \frac{3^2}{3^2 - 1} \cdot \frac{5^2}{5^2 - 1} \cdot \frac{7^2}{7^2 - 1} \cdots .$$

Moreover, Euler showed that if the squares in the foregoing series and product are replaced by fourth powers, the result is $\pi^4/90$, and if they are replaced by sixth powers the result is $\pi^6/945$. In fact, Euler showed that if the squares are replaced by any even exponent $2n$, the result is equal to π^{2n} multiplied by a rational number.

The number π appears unexpectedly in many situations that have nothing to do with the circle or its circumference. It shows up in formulas in engineering problems, and in many problems related to random events. Its connection with probability is discussed further in Section 6 of this workbook.

3. *Early history of π*



The existence of a fixed ratio between the circumference of a circle and its diameter was known to many ancient civilizations. Both the Babylonians and the Egyptians knew that this ratio was greater than 3. Babylonian clay tablets have been found in which the circle ratio is given as 3 1/8, or 3.125. The Egyptians gave a slightly different value, obtained by comparing the area of a circular disk with the square of its diameter. In an Egyptian papyrus, written before 1700 B.C., the area of a circular disk is found by squaring eight ninths of its diameter. If we take the diameter to be 2, the area is π, and the Egyptian rule gives

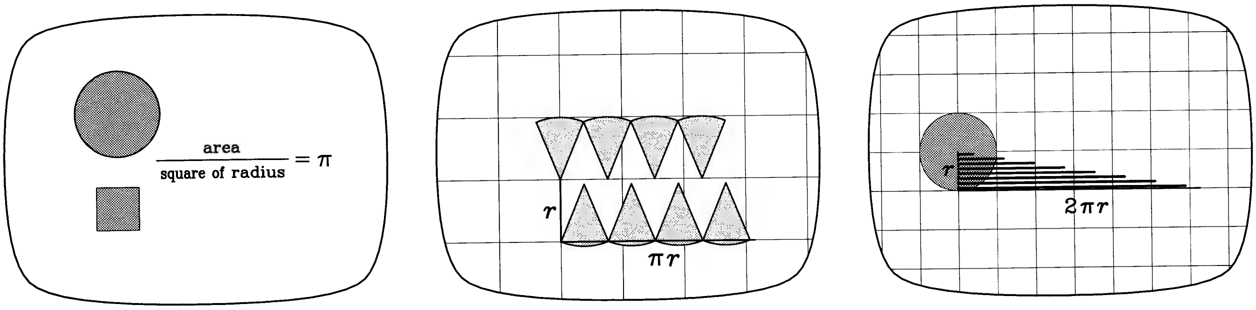
$$\pi = (2 \times \frac{8}{9})^2 = (\frac{16}{9})^2 = \frac{256}{81} = 3.1605.$$

The old testament describes a circular basin or "molten sea," made by Hiram of Tyre in Solomon's temple. The basin is described as being ten cubits across and thirty cubits around, which would make π equal to 3. However, at that point in history it was widely known that π was greater than 3, and there is no reason to believe that the biblical text was intended to be anything more than a casual description.

Although many ancient civilizations had observed through measurement that the circle ratio is the same for circles of different sizes, the Greeks were the first to explain why. It's a simple property of similar figures. Suppose you start with a circle of diameter d and circumference C. If you expand or contract the circle by a scaling factor s, you get a circle of diameter sd and circumference sC. The ratio of the new circumference to the new diameter is sC/sd. The scaling factor s cancels and we get C/d, the same ratio as for the original circle.

The ancient Greeks were probably also the first to realize that π, like $\sqrt{2}$, is a number quite unlike any of the whole numbers or rational numbers (ratios of whole numbers) that they used in their mathematics. Although the Greeks were able to prove that $\sqrt{2}$ is irrational (not a rational number) they did not succeed in doing the same for π, and there is no evidence that they attempted to do so. A proof that π is irrational was not found until many centuries later when formulas were discovered that made it possible to evaluate π without using geometric figures.

4. *A discovery of Archimedes*

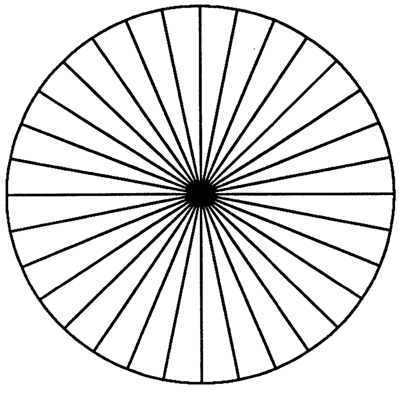


As noted earlier, if the unit circular disk has circumference C and area A, then the circular disk of radius r has circumference Cr (because lengths get multiplied by r) and area Ar^2 (because areas get multiplied by r^2). In other words, the two constants C and A tell us how to determine the circumference and area of any circular disk in terms of its radius. Therefore they represent two fundamental constants of nature. Archimedes (287-212 B.C.), the greatest mathematician of ancient times, was the first to prove that the two constants C and A enjoy a simple relationship:

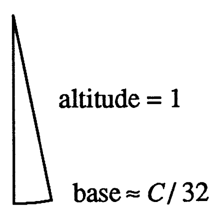
$$C = 2A.$$

In this section we shall see how he proved this. The constant C is, of course, 2π (because of the definition of π), which means that $A = \pi$.

The method Archimedes used to prove that $C = 2A$ is quite simple. Imagine a unit circular disk divided into a large number of equal slices made with radial cuts from the center. Figure (a) below shows a disk cut into 32 equal slices. The area of the entire disk is 32 times the area of one slice. Each slice in turn, is very nearly a triangle with altitude 1 and base $C/32$, as suggested by figure (b). The area of such a triangle is one-half the base times the altitude, or $C/64$. The area of the circular disk is 32 times as great, or $C/2$. In other words, by approximating each small slice by a triangle, we see that A is nearly $C/2$. Archimedes refined this idea, taking an arbitrary number of slices and estimating the error made in the approximation. He then showed that the error approaches 0 as the number of slices becomes arbitrarily large, and thus deduced that $A = C/2$, or $C = 2A$.



(a) A disk divided into 32 equal slices.



(b) Each small slice is nearly a triangle.

Exercises involving the circumference and area of a circular disk

1. A satellite makes a circular orbit around the earth at a constant altitude. What change in altitude is required to lengthen each orbit by one mile?

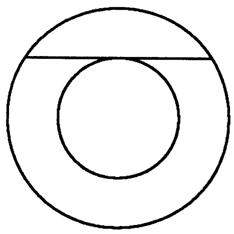
2. The radius of a circular disk is increased from r to $r + h$.

(a) By how much does the circumference change? By how much does the area change?

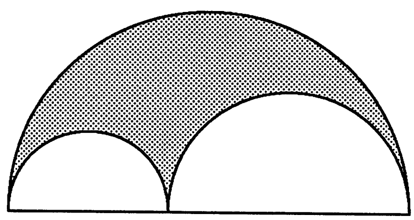
(b) What value of h is required to double the original circumference? To double the original area?

3. The horizontal segment in Figure (a) below has length $2a$. Prove that the area of the region between the two circles is πa^2.

4. In Figure (b), semicircles of diameters a and b are drawn inside a circle of diameter $a + b$. Prove that the shaded region (called an *arbelos* or *cobbler's knife*) has area $\pi ab/4$.



(a) Exercise 3.



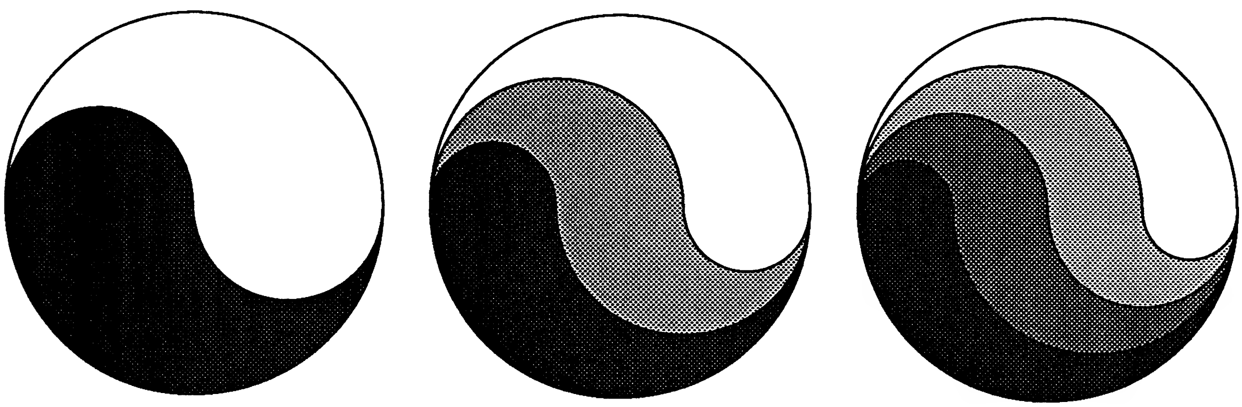
(b) Exercise 4.

5. In each of the following diagrams the horizontal diameter of the large circle is divided into equal segments and all curves shown are made from semicircular arcs. In Figure (a), it is clear that the large circular disk is divided into two regions of equal area and equal perimeter.

(a) Prove that the three regions in (b) have equal areas and equal perimeters.

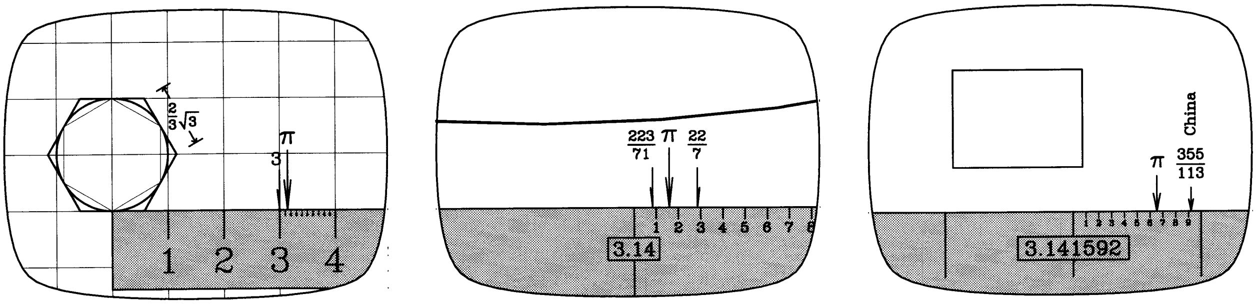
(b) Prove that the four regions in (c) have equal areas and equal perimeters.

(c) State and prove a generalization suggested by these diagrams.



(a) (b) (c)

5. *Computation of π*



Because π was recognized as a fundamental constant of nature, people tried for centuries to determine its numerical value accurately. The first serious attempt was made by Archimedes, who obtained approximate values of π by comparing the circumference of a circle with the perimeters of inscribed and circumscribed regular polygons. For a unit circle, an inscribed hexagon consists of six equilateral triangles with edge 1. Its perimeter, 6, is less than the circumference of the circle, 2π, so $3 < \pi$. In Exercise 1 below, the reader is asked to show that the circumscribed hexagon consists of six equilateral triangles with edge $2\sqrt{3}/3$. Its perimeter is greater than the circumference of the circle, so $2\pi < 4\sqrt{3}$, or $\pi < 2\sqrt{3}$. This gives the following upper and lower bounds:

$$3 < \pi < 2\sqrt{3}.$$

In modern decimal notation, these estimates can be written as

$$3 < \pi < 3.464.$$

Archimedes approximated $\sqrt{3}$ by the rational number 265/153, which is remarkably close to $\sqrt{3}$. In fact, $(265/153)^2 = 70225/23409 = 2.999146$. Using the approximation 265/153 for $\sqrt{3}$ we find $2\sqrt{3}$ is approximated by $530/153 = 3.4640523$.

Archimedes kept doubling the number of sides, always replacing the square roots of integers by remarkably accurate rational approximations. For example, a 12-sided polygon produced the square root of 349450, which he approximated by 591 1/8. It is not known how he arrived at these extraordinary rational approximations to square roots. When he reached a polygon of 96 sides his rational approximations led to the inequalities

$$3\,\frac{10}{71} < \pi < 3\,\frac{1}{7},$$

which, in decimal form, state that $3.14084507 < \pi < 3.14285714$. This gives the value 3.14, correct to two decimal places. It was an incredible achievement because it was obtained without the use of algebraic symbols, Arabic numerals, or decimal notation that are in common use today.

What is more important, Archimedes made a significant contribution to mathematical thought by proving that the difference between the inner and outer perimeters becomes arbitrarily small as the doubling process is repeated indefinitely. When he showed that both the inner and outer perimeters approach the same number as the number of doublings becomes infinite, Archimedes made the first significant use of the limit of an infinite sequence of numbers, an idea that eventually developed into the methods of calculus.

Later investigators obtained better rational approximations to π by using polygons with more sides. An impressive Chinese calculation with a polygon of more than 3,000 sides gave five decimals for π. The Chinese also found a simple fraction, 355/113, which differs from π by less than 0.0000003. They held the world's record for more than a thousand years until the use of Arabic numerals and decimal notation provided more efficient ways of doing arithmetic. The rational approximation 355/113 was rediscovered in the 16th century by the Dutch engineer Adriaan Anthoniszoon. In the same century, another Dutchman, Adriaen van Rooman, used Archimedean polygons with 2^{30} sides to obtain 15 decimal places for π. A few years later Ludolph van Ceulen, professor of mathematics and military science at the University of Leyden, obtained 20 decimals from a polygon with 60×2^{29} sides, and later extended this result to 35 decimal places. The Germans were so impressed by this calculation that for many years they called π the Ludolphine number.

When infinite series and trigonometric functions became widely known, formulas were discovered that made it possible to approximate π without using geometric diagrams. In 1706 John Machin used an infinite series for the arctangent function to calculate π to a hundred decimal places. A new record for hand calculation of π was set in 1874 by William Shanks, who cranked out 707 decimal places. Unfortunately, there was an error after the 526th place, but the error wasn't discovered until 1945 when the calculation was repeated on an electronic calculating machine.

In the twentieth century, high-speed electronic computers were coupled with new mathematical methods that enabled computers to do arithmetic with very long numbers. More than one billion decimals of π were known in 1989, and plans were made to extend the calculation even further.

What's the point of grinding out all those decimals for π? After all, fourteen decimals are all you need to figure out the circumference of the earth's equator with an error less than one thousandth of a millimeter. So, why go to the trouble and expense of hooking up giant supercomputers to calculate a billion decimals of π? One reason is to test the architecture of supercomputers. Computing a million digits of π gives large computers a thorough workout, and provides a good measure of the computer's overall efficiency. Occasionally unexpected quirks are revealed, peculiar to the machine being used. The calculation also checks software and programs for speed and accuracy.

Sometimes these calculations reveal more about π itself. For a long time people wondered if π was an exact fraction, like 22/7. A ratio of two integers, called a rational number, always has repeating patterns in its decimal form. For example, the decimal version of 22/7 is 3.142857142857..., with the block of digits 142857 repeated indefinitely. As people calculated more and more decimals for π they searched for patterns that repeat over and over, but none were found. Finally, in the eighteenth century, it was shown that none would ever be found. In 1761 the German mathematician Johann Lambert used a continued fraction for the trigonometric tangent of an angle to show conclusively that π is irrational, that is, π is not the ratio of two integers.

The square root of 2, like π, is also irrational, but there is a significant difference between π and $\sqrt{2}$ that has to do with geometry. A line segment of length $\sqrt{2}$ can be constructed from a segment of unit length by Euclidean methods, that is, by using only an unmarked straightedge and a compass. But a segment of length π cannot be constructed in this way. It is known that every number that can be constructed by Euclidean methods is a root of a polynomial equation with integer coefficients. The number $\sqrt{2}$ is an example because it is a root of the equation

$$x^2 - 2 = 0.$$

Real numbers, like $\sqrt{2}$, that are roots of polynomial equations with integer coefficients are called *algebraic numbers*. Numbers like π that are not algebraic are called *transcendental numbers*. In 1882 another German mathematician, F. Lindemann, proved that both π and $\sqrt{\pi}$ are transcendental.

Lindemann's discovery settled a problem posed by the ancient Greeks. They asked whether it was possible to construct, by Euclidean methods, a square equal in area to a given circle. The problem became known as that of "squaring the circle." If the radius of the circle is 1, its area is π, so each side of the proposed square would have length $\sqrt{\pi}$. Lindemann's proof that $\sqrt{\pi}$ is transcendental proves that it is impossible to square the circle. This of course does not mean that the proposed square does not exist. It does exist, but it cannot be constructed in the manner required by the Greek geometers, using only straightedge and compass.

Exercises involving estimates for calculating π

1. (a) Calculate the areas of regular hexagons inscribed and circumscribed about a unit circular disk and thereby deduce the inequalities

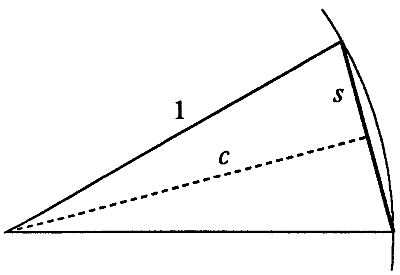
$$\frac{3}{2}\sqrt{3} < \pi < 2\sqrt{3}.$$

(b) Calculate the perimeters of the regular hexagons in part (a) and deduce the inequalities

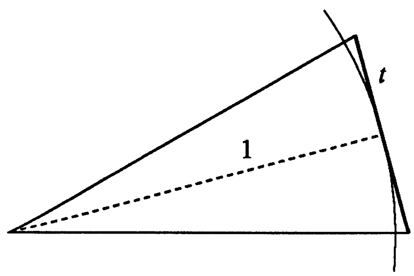
$$3 < \pi < 2\sqrt{3}.$$

2. (a) Suppose a regular polygon with n sides is inscribed in a unit circle. The polygon consists of n congruent copies of the isosceles triangle shown in (a). The triangle can be bisected into two right triangles with legs of length s and c, and hypotenuse of length 1, as shown. Prove that the inscribed polygon has perimeter $2ns$ and area ncs.

(b) Suppose a regular polygon with n sides is circumscribed about a unit circle. The polygon consists of n congruent copies of the isosceles triangle shown in (b). The triangle can be bisected into two right triangles with legs of length 1 and t, as shown. Prove that the circumscribed polygon has perimeter $2nt$ and area nt.



(a) Part of the inscribed polygon.



(b) Part of the circumscribed polygon.

(c) Show that the following inequalities for π are obtained by comparing the circumference of the circle with the perimeters of the inscribed and circumscribed polygons,

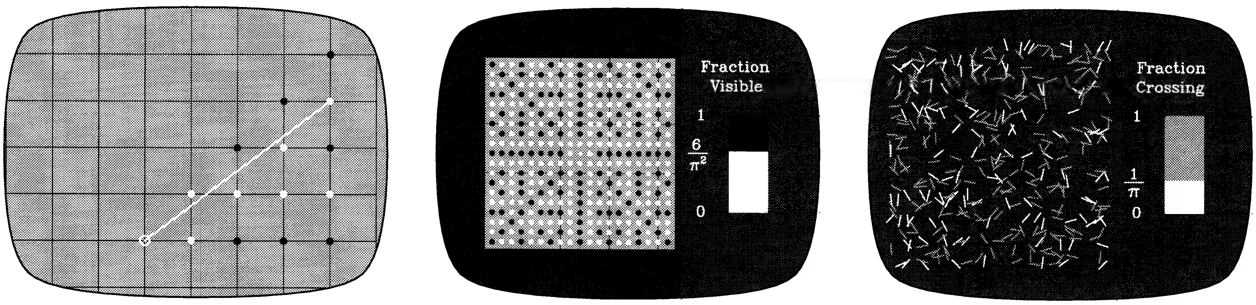
$$ns < \pi < nt,$$

while the following inequalities come from comparison of areas,

$$ncs < \pi < nt.$$

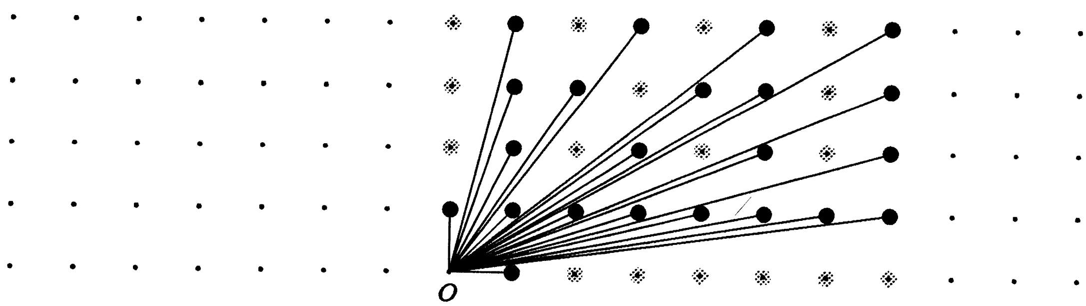
Note that the upper bounds for π obtained by using perimeters and areas are the same, but, because $c < 1$, the lower bound using perimeters is better than that obtained by using areas.

6. *Further uses of π*



The number π occurs not only in the formulas for the circumference and area of a circle, but also in formulas for volumes and areas of circular objects such as cylinders, cones, or spheres. Legend has it that when Archimedes discovered that π enters into the formulas for the area and volume of a sphere, he was so impressed that he wanted these facts recorded on his tombstone. He would have been even more impressed to learn that π also appears in many situations that have nothing to do with circles. For example, $\sqrt{\pi}$ is related to the area of a bell-shaped curve, called a Gaussian curve, which occurs in many problems in probability and statistics. This curve is used by life insurance companies in constructing mortality tables, and it also occurs in engineering problems concerned with radiation and heat flow. This section describes two problems in probability that lead to expressions involving the reciprocal of π and of π^2.

Cover a plane with identical square tiles of unit length. The vertices of the squares are called *lattice points*. Choose a convenient lattice point and call it O, for *origin*. Now choose another lattice point P, and draw the segment joining O to P. If no other lattice points lie on this segment, we say P is visible from the origin. In the following diagram, lattice points marked with large dots are visible from the origin, but those that are shaded are not.



If you draw a large square centered at the origin and count lattice points, you'll find that about 60% of the lattice points are visible from the origin. As the square expands to cover the whole plane the fraction of visible points fluctuates somewhat, but it can be shown that this fraction approaches the limiting value $6/\pi^2$. This surprising result is often described as follows:

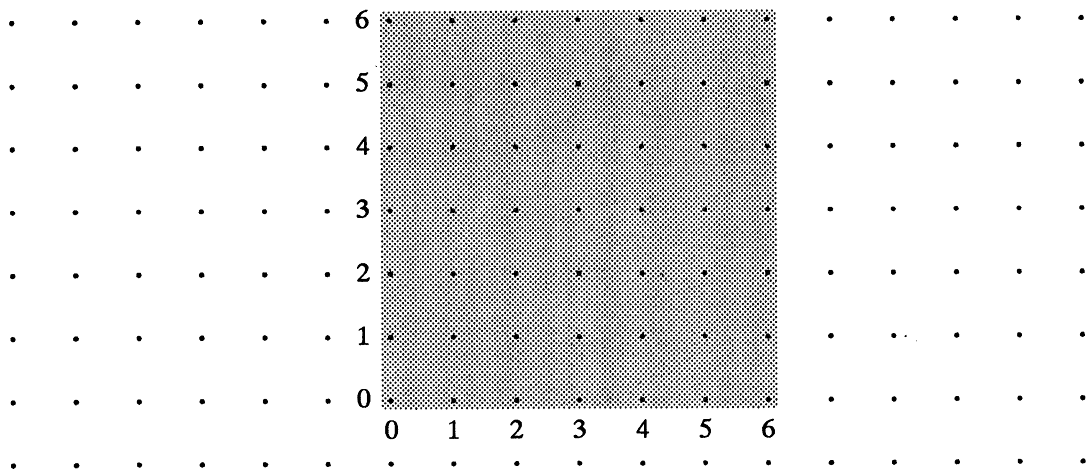
A lattice point chosen at random has the probability $6/\pi^2$ *of being visible from the origin.*

The same result can be described in terms of integers. Rectangular coordinates axes can be drawn intersecting at the origin so that each point in the plane is assigned a pair of coordinates (x, y). The lattice points are those points with integer coordinates. It can be shown that a lattice point (x, y)

is visible from the origin if, and only if, the integers x and y have no prime factor in common. Two integers with no prime factor in common are said to be *relatively prime*. The foregoing result on visibility of lattice points can be restated as follows:

If two integers are chosen at random, the probability that they are relatively prime is $6/\pi^2$.

A proof of this result requires advanced mathematics, but the following discussion indicates how the number $6/\pi^2$ arises. If x is a given integer and p is a given prime, divide x by p to obtain a quotient and a remainder, where the remainder is one of the numbers $0, 1, 2, \ldots, p - 1$. The remainder depends on x and p and can be denoted by the symbol x_p. Note that the remainder is 0 if p is a prime factor of x; otherwise the remainder is positive. Now choose another integer y, divide it by the same prime p and get a corresponding quotient and remainder y_p. There are p possible values for x_p and p possible values for y_p, and therefore p^2 possible values for the pair of numbers (x_p, y_p). When plotted as a lattice point, each (x_p, y_p) falls on one of the p^2 lattice points with coordinates running from 0 to $p - 1$. An example with $p = 7$ is shown here.



One of these p^2 lattice points is the origin, and the lattice point (x_p, y_p) falls on the origin if, and only if, both x and y are divisible by p. So, the probability that both x and y are divisible by the same prime p is $1/p^2$. Consequently, the probability that at least one of x or y is *not* divisible by p is equal to $1 - 1/p^2$, or $(p^2 - 1)/p^2$. It seems reasonable to assert that the probability that *no* prime divides both x and y is the product of the probabilities $(p^2 - 1)/p^2$ taken over all primes p. This is an infinite product that can be written as follows:

$$\frac{2^2 - 1}{2^2} \cdot \frac{3^2 - 1}{3^2} \cdot \frac{5^2 - 1}{5^2} \cdot \frac{7^2 - 1}{7^2} \cdots .$$

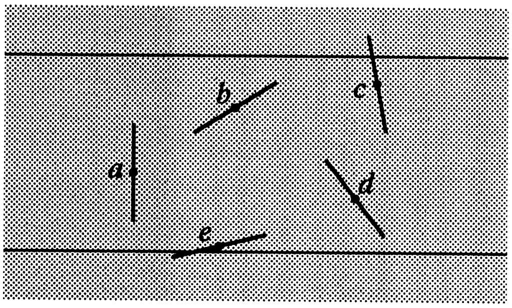
Each factor in this product is the reciprocal of the corresponding factor in Euler's infinite product for $\pi^2/6$ mentioned earlier:

$$\frac{\pi^2}{6} = \frac{2^2}{2^2 - 1} \cdot \frac{3^2}{3^2 - 1} \cdot \frac{5^2}{5^2 - 1} \cdot \frac{7^2}{7^2 - 1} \cdots .$$

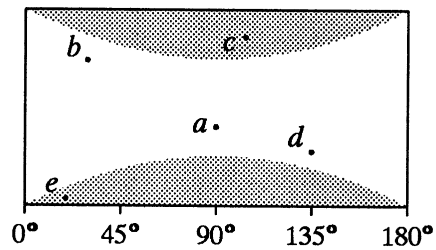
The reciprocal of this product, $6/\pi^2$, is the probability that two integers chosen at random will be relatively prime.

The Buffon needle problem

A celebrated problem in geometrical probability, known as the *Buffon needle problem,* also involves π. If a needle of length L (say two inches) is dropped at random on a floor made of planks of width d greater than L (say four inches wide), what is the probability that the needle will fall across a crack between two planks? If the needle is exactly half the width of the planks, the answer turns out to be $1/\pi$, or about 0.318. In other words, if a large number of needles are dropped at random on the floor, about 32% of the needles will either touch or cross a crack between two planks. The solution of this problem requires calculus, but we can get an idea of how it is arrived at by referring to the following diagram.



(a) Possible positions of the needle.



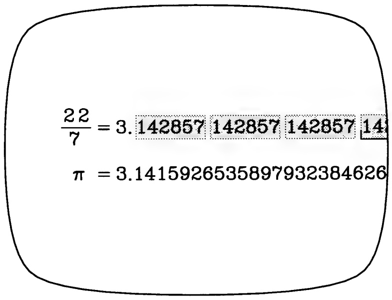
(b) Alternate representation of positions.

We specify the position of each needle by noting the position of its midpoint and the angle that the needle makes with a given crack. Figure (a) shows five possible positions. The rectangle in (b) shows another way to indicate the various positions of the needle. The height of the rectangle represents the width of the plank, and the base represents the angle the needle makes with the bottom of the plank. The angle can be anything from 0 to 180°. Each point of the rectangle represents a possible position in which a needle can fall. Points in the shaded region correspond to positions of needles that lie across a crack. Assuming all possible positions of the needle are equally likely, the probability of the needle falling on a crack is the ratio of the area of the shaded region divided by the area of the rectangle. This ratio can be found by using calculus and turns out to be $2L/(\pi d)$, where L is the length of the needle and d is the width of the plank. And when $d = 2L$, the ratio is $1/\pi$.

Using random numbers to approximate π

Here's an experiment you can do with random numbers to estimate the value of π. If you don't have access to a device that generates random numbers you can use a telephone book. Pick any telephone number, retain only the last four digits, and insert a decimal point in front of them. For example, if the telephone number is 356-3759, we get .3759. Call this number x. Now choose another telephone number and let y be the four-digit decimal obtained. Plot the point with rectangular coordinates (x, y) on graph paper. Because both x and y are between .0000 and .9999, this point lies somewhere in a unit square with opposite vertices (0, 0) and (1, 1). Let r denote the distance of the point (x, y) from the origin. If $r \le 1$, the point (x, y) lies in the circular disk of radius 1 with center at (0, 0). If $r > 1$, the point lies outside the disk. Now take a large sample of pairs of telephone numbers, say n pairs, where n is at least 200. For each pair, plot the corresponding point (x, y) and count the number of pairs for which the distance $r \le 1$. Call this number k. If you try this experiment you will find that the ratio $4k/n$ is a reasonable approximation to π. Try the experiment with at least 200 pairs and explain why you think the ratio $4k/n$ should be approximately equal to π.

7. *Recap*



The number π is defined to be the ratio of the circumference to the diameter of any circle. A fundamental property of similar figures states that this ratio is the same for all circles, regardless of their size. Similarity also shows that the ratio of the area of a circular disk to the square of its radius is also constant. Archimedes proved that this constant ratio is the same number π. So, π appears in the formulas $2\pi r$ and πr^2 for calculating the circumference and area of a circular disk of radius r. Archimedes went on to obtain the now familiar formulas $4\pi r^2$ and $4\pi r^3/3$ for the area and volume of a sphere of radius r. We are no longer surprised to find π in formulas for round geometric objects, but it certainly is surprising to find π appearing in contexts that seem to have no relation to geometry. It can be represented by an infinite product of simple fractions, it is closely related to various infinite sums of reciprocals of integers, and it appears in interesting probability problems. Because the number π is so fundamental, people have tried for centuries to determine its numerical value accurately.

Advances in the computation of π have been markers of significant progress in the history of mathematics. Archimedes gave the first known proof concerning the value of π when he compared perimeters of polygons inside and outside a circle. He showed that π, like any real number, can be approximated to any degree of accuracy by rational numbers. His investigation of π was the first use of the concept of the limit of an infinite numerical sequence, a profound idea that formed the basis for the development of calculus several centuries later.

The Archimedes estimate, 22/7, which gives π correct to two decimals, was an extraordinary accomplishment. It was improved 700 years later by the Chinese value 355/113, which gives π correct to six decimals. Better rational approximations to π accompanied major advances in mathematics, such as the adoption of Arabic numerals, decimal notation, the use of infinite series, the methods of calculus, and new methods developed in the twentieth century to exploit the technology of super computers. Important mathematical ideas concerned with irrational and transcendental numbers grew out of investigations concerning the nature of the number π.

People have calculated more than one billion digits of π because of the human desire to do something that's never been done before. It's a challenge to the human spirit, like climbing a mountain or traveling to the outer planets. When George Mallory was asked why he wanted to climb Mt. Everest he replied, "Because it's there." Well, π is certainly there. Like the outer planets, it's built into the fabric of our physical universe, and it will always be explored.

Pi Miscellany

This brief workbook contains only a small sample of the wealth of material written about π. Much more can be found in a charming little book entitled *A History of Pi*, by Petr Beckmann (2nd edition, 1971, Golem Press, Box 1342, Boulder, Colorado 80302) and also in a comprehensive two-part article entitled "The Ubiquitous π," by Dario Castellanos, published in *Mathematics Magazine* (Part I in Vol. 61, No. 2, April 1988, pp. 67-98; Part II in Vol. 61, No. 3, June 1988, pp. 148-163). Each of these references contains a lengthy bibliography of books and articles about π. We conclude this workbook with some miscellaneous items related to π.

The rough estimate of 3 for π mentioned in the Bible is also found in the following passage from Jules Verne's *Twenty Thousand Leagues under the Sea*:

> The Nautilus was stationary, floating near a mountain which formed a sort of quay. The lake then supporting it was a lake imprisoned by a circle of walls, measuring two miles in diameter and six in circumference.

The Hindu mathematician Brahmagupta (born 598 A.D.) gave the approximate value $\pi \approx \sqrt{10}$. Apparently he applied Archimedes' method of successive doublings to a circle of diameter 10, obtaining the respective values $\sqrt{965}$, $\sqrt{981}$, $\sqrt{986}$, and $\sqrt{987}$ for the perimeters of inscribed polygons with 12, 24, 48 and 96 sides. This led him to believe that continued doublings would converge to the limiting value $\sqrt{1000}$ for the perimeters, giving $\sqrt{10}$ for π. The error in this approximation is revealed by the infinite series

$$10 - \pi^2 = \frac{1}{1^3 2^3} + \frac{1}{2^3 3^3} + \frac{1}{3^3 4^3} + \frac{1}{4^3 5^3} + \cdots,$$

which can be deduced from Euler's formula for the sum of the squares of the reciprocals of the positive integers (mentioned on page 18).

Lambert showed that π is equal to the limit of a sequence of fractions, the first three terms of which are

$$x_1 = \frac{22}{7}, \quad x_2 = x_1 - \frac{1}{7\cdot 106} = \frac{333}{106}, \quad x_3 = x_2 + \frac{1}{106\cdot 113} = \frac{355}{113}.$$

An easy way to remember the approximation 355/113 is to write the first three odd numbers twice: 1 1 3 3 5 5, and insert the symbol for long division in the center of this sequence as follows:

$$113\overline{)355}\ .$$

Note that 355/113 = 3 + 16/113. Like all rational numbers, the fraction 16/113 has a repeating pattern in its decimal version--in this case, a block of 112 digits:

16/113 = 0.1415929203 5398230088 4955752212 3893805309 7345132743 3628318584
0707964601 7699115044 2477876106 1946902654 8672566371 68.

The first 200 decimal places of π are listed here:

3.1415926535 8979323846 2643383279 5028841971 6939937510 5820974944 5923078164
0628620899 8628034825 3421170679 8214808651 3282306647 0938446095 5058223172
5359408128 4811174502 8410270193 8521105559 6446229489 5493038196 .

INDEX

Project MATHEMATICS!

COMPUTER-ANIMATED MATHEMATICS VIDEOTAPES

The Story of Pi is part of a series of modules designed to use computer animation to help instructors teach basic concepts in mathematics. Each module consists of a videotape, about 20 minutes in length, and a workbook to guide students through the video, elaborating on the important ideas. The modules are used as support material for existing courses in high school and community college classrooms, and may be copied without charge for educational use.

Based at the California Institute of Technology, Project *MATHEMATICS!* has attracted as partners the departments of education of 32 states in a consortium whose members reproduce and distribute the videotapes and written materials to public schools. The project is headed by Tom M. Apostol, professor of mathematics at Caltech and an internationally known author of mathematics textbooks. Co-director of the project is James F. Blinn, one of the world's leading computer animators, who is well known for his Voyager planetary flyby simulations. Blinn and Apostol worked together previously as members of the academic team that produced *The Mechanical Universe*, the award-winning physics course for television also developed at Caltech.

The first videotape produced by *Project MATHEMATICS!*, entitled *The Theorem of Pythagoras*, has been distributed to thousands of classrooms nationwide. It received first-place awards at five international competitions: a Gold Medal at the 1988 International Film & TV Festival of New York; a Gold Apple Award at the 1989 National Educational Film & Video Festival, Oakland, California; a Blue Ribbon, Best of Category, at the 1989 American Film and Video Festival, Chicago, Illinois; an Electra Certificate, Best of Category, at the 1989 Birmingham International Educational Film Festival; and a Gold Cindy at the 1989 Cindy Competition, Los Angeles, California. *The Story of Pi* was awarded a Gold Apple Award at the 1990 National Educational Film & Video Festival.

Information concerning the project can be obtained by writing to the project director at the address on the title page of this booklet. Copies of the videotape and workbook on *The Story of Pi* or on *The Theorem of Pythagoras* can be obtained from the Caltech Bookstore, 1-51 Caltech, Pasadena, CA 91125. (Tel: 818-356-6161) They can also be obtained from the National Council of Teachers of Mathematics or the Mathematical Association of America at the addresses given below.

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